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Forwarding index of cube-connected cycles[☆]

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ABSTRACT

For a given connected graph G of order v, a routing R in G is a set of $v(v-1)$ elementary paths specified for every ordered pair of vertices in G. The vertex (resp. edge) forwarding index of G is the maximum number of paths in R passing through any vertex (resp. edge) in G. Shahrokhi and Székely [F. Shahrokhi, L.A. Székely, Constructing integral flows in symmetric networks with application to edge forwarding index problem, Discrete Applied Mathematics 108 (2001) 175–191] obtained an asymptotic formula for the edge forwarding index of n-dimensional cube-connected cycle CCC_n as $\frac{5}{4}n^2 2^n(1-o(1))$. This paper determines the vertex forwarding index of CCC_n as $\frac{7}{4}n^2 2^n(1-o(1))$ asymptotically.

1. Introduction

A routing R in a connected graph G of order v is a set of $v(v-1)$ elementary paths $R(x, y)$ specified for every ordered pair (x, y) of vertices of G. A routing R is said to be minimal if every path $R(x, y)$ in R is a shortest path from x to y in G. To measure the efficiency of a routing deterministically, Chung et al. [3] introduced the concept of the forwarding index of a routing.

The load $\xi(G, R, x)$ of a vertex x (resp. the load $\pi(G, R, e)$ of an edge e) with respect to R is defined as the number of paths specified by R going through x (resp. e). The parameters

$$\xi(G,R) = \max_{v\in V(G)} \xi(G,R,v) \quad \text{and} \quad \pi(G,R) = \max_{e\in E(G)} \pi(G,R,e)$$

are defined as the vertex forwarding index and the edge forwarding index of G with respect to R, respectively; and the parameters

$$\xi(G) = \min_{R} \xi(G,R) \quad \text{and} \quad \pi(G) = \min_{R} \pi(G,R)$$

are defined as the vertex forwarding index and the edge forwarding index of G, respectively.

The original study of forwarding indices was motivated by the problem of maximizing network capacity, see [3]. Minimizing the forwarding indices of a routing will result in maximizing the network capacity. Thus, it becomes very significant to determine the vertex and the edge forwarding indices of a given graph, see [14] for details. However, Saad [12] found that for an arbitrary graph determining its vertex forwarding index is NP-complete even if the diameter of the graph is two. Even so, the forwarding indices of many well-known networks have been determined by several researchers, see, for example, [1,3,4,6–9,12,16–18].

In this paper, we consider the n-dimensional cube-connected cycle CCC_n. Shahrokhi and Székely [13] obtained an asymptotic formula $\pi(CCC_n) = \frac{5}{4}n^2 2^n(1-o(1))$. We determine $\xi(CCC_n) = \frac{7}{4}n^2 2^n(1-o(1))$ asymptotically. The proof of our result is in Section 4. In Section 2, we recall the definition and some properties of CCC_n. In Section 3, we show a minimal routing of CCC_n.

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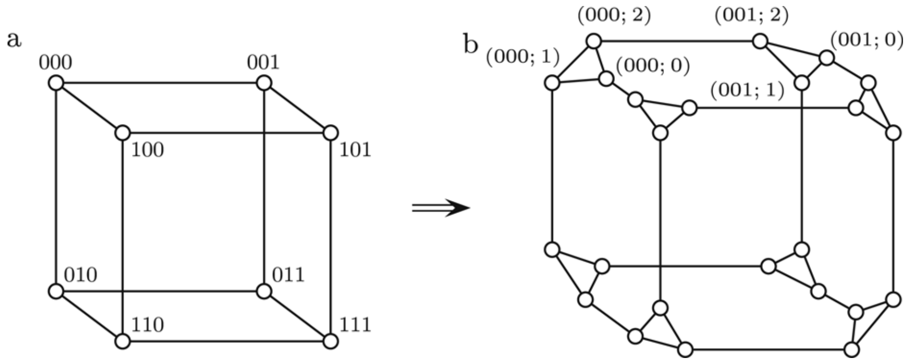

Fig. 1. A cube-connected cycle CCC_3 constructed from Q_3.

2. Cube-connected cycles

In this section, we recall the definition and some properties of the cube-connected cycle. We follow the standard terminology and notation of Xu [15].

The n-dimensional cube, or hypercube, denoted by $Q_n = (V(Q_n), E(Q_n))$, has vertices

$$V(Q_n) = \{x_0x_1 \cdots x_{n-1} : x_i \in \{0, 1\},\ 0 \le i \le n-1\},$$

and two vertices $x = x_0x_1 \cdots x_{n-1}$ and $y = y_0y_1 \cdots y_{n-1}$ are linked by an edge if and only if they differ in exactly one coordinate, i.e.,

$$xy \in E(Q_n) \Leftrightarrow \sum_{i=0}^{n-1} |x_i - y_i| = 1.$$

The graph shown in Fig. 1(a) is Q_3.

We call an edge linking two vertices $x_0 \cdots x_{i-1}x_ix_{i+1} \cdots x_{n-1}$ and $x_0 \cdots x_{i-1}\overline{x_i}x_{i+1} \cdots x_{n-1}$ an i-dimensional edge in Q_n, where $\overline{x_i} = 1 - x_i$.

The n-dimensional cube-connected cycle, denoted by CCC_n, is constructed from Q_n by replacing each of its vertices with a cycle $C_n = (0, 1, \ldots, n-1)$ of length n. The i-dimensional edge incident with a vertex of Q_n is connected to the ith vertex of C_n. For example, CCC_3 shown in Fig. 1(b) is constructed from Q_3.

By modifying the labeling scheme of Q_n, we can represent each vertex of CCC_n by a pair $(x; i)$, where $x \in V(Q_n)$ and $i \in V(C_n)$. Precisely, the vertex set of CCC_n is

$$V = \{(x; i) : x \in V(Q_n), 0 \le i \le n-1\},$$

where $x \in V(Q_n)$ is called the cubic coordinate and i $(0 \le i \le n-1)$ the cyclic coordinate of the vertex $(x; i)$. Two vertices $(x; i)$ and $(y; j)$ are linked by an edge in CCC_n if and only if either:

(i) $x = y$ and $|i - j| \equiv 1 \pmod n$, or
(ii) $i = j$ and x differs from y in exactly the ith coordinate.

We call edges of the type (i) *cyclic edges* and edges of the type (ii) *cubic edges*.

It is quite apparent from its construction that CCC_n is a 3-regular and 3-connected graph of order $n2^n$. It is also clear that CCC_n contains Hamilton cycles. In fact, Germa et al. [5] investigated all lengths of cycles contained in CCC_n. Krishnamoorthy and Krishanmirthy [10] proved that CCC_n has diameter $\lfloor \frac{1}{2}(5n-2) \rfloor$. Furthermore, Carlsson et al. [2] showed that CCC_n is a Cayley graph and, hence, is vertex-transitive, also see Xu [14] for details.

The cube-connected cycle, first formalized and extensively studied by Preparata and Vuillemin [11], has almost all desirable features of the hypercube and overcomes the drawbacks of the hypercubes. It can be used not only as an interconnection pattern of general purpose parallel processing systems, but also in the layout of many specialized large scale integrated circuits. The cube-connected cycle provides a communication pattern to implement some algorithms for efficiently solving a large class of problems that include Fast Fourier transform, sorting, permutations, and derived algorithms, see [11] for details. Thus, the cube-connected cycle is a feasible substitute for the hypercube network.

3. Shortest paths in CCC_n

Let o be the vertex $00 \cdots 0$ in Q_n and $C_n = (0, 1, \ldots, n-1)$ be a cycle of length n. For an arbitrary vertex $(x; j)$ in CCC_n, the shortest paths from $(o; 0)$ to $(x; j)$ in CCC_n are closely related to $(0, j)$-walks on C_n. Before exploring such relations, we will introduce some useful notations.

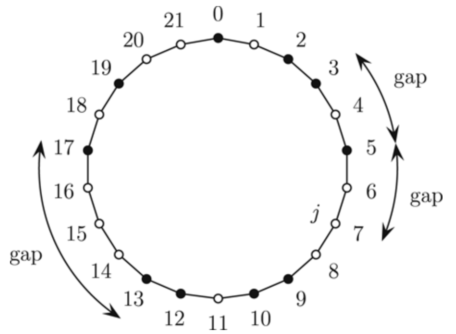

Fig. 2. C_{22} and gaps, where $j = 7, \ell_1 = 2, \ell_2 = 4, S = \{0, 2, 3, 5, 9, 10, 12, 13, 17, 19\}$.

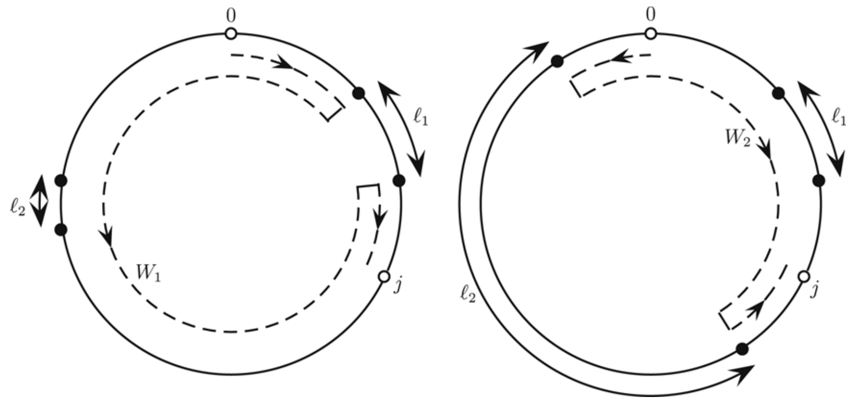

Fig. 3. Two forms of $(0, j)$-walks.

Let S be a subset of $\{0, 1, \dots, n-1\}$. For every vertex j in C_n, we can partition C_n into two paths: $P_1 = (0, 1, \dots, j)$ and $P_2 = (j, j+1, \dots, n-1, 0)$. When $j = 0$, which we call the degenerated case, we define path P_1 as a path of length 0, and path P_2 a closed path of length n.

We then define *gap* of path P_1 and P_2, respectively. When $1 \le j \le n-1$, we define the *gap* $\ell_1(j, S)$ of P_1, with respect to j and S, as the maximum length of sub-paths in P_1 divided by vertices in $S \cap \{1, \dots, j-1\}$ (there might be multiple sub-paths achieving the maximum length); when $S \cap \{1, \dots, j-1\} = \emptyset$, then it is j. For the degenerated case $j = 0$, we simply define that the gap is 0. In the same way, for every $j(0 \le j \le n-1)$, we can define the *gap* $\ell_2(j, S)$ of P_2 as the maximum length of sub-paths in P_2 divided by vertices in $S \cap \{j+1, \dots, n-1\}$; and when $S \cap \{j+1, \dots, n-1\} = \emptyset$, then it is $n-j$. (Note that here view "+" as "mod n"; so when $j = n-1$, we have $j+1 = 0$.) In the following context without ambiguity, we write ℓ_1 and ℓ_2 for $\ell_1(j, S)$ and $\ell_2(j, S)$, respectively. Define $m(j, S)$ as the length of a shortest $(0, j)$-walk in C_n that traverses all vertices in S.

For example, the graph shown in Fig. 2 is a cycle $C_{22}, j = 7, \ell_1 = 2, \ell_2 = 4, S = \{0, 2, 3, 5, 9, 10, 12, 13, 17, 19\}$ depicted by solid vertices.

The concept of the gap, first introduced by Shahrokhi and Székely [13], can be used to express $m(j, S)$. By the symmetry of j and $n-j$, we can assume $0 \le j \le \lfloor n/2 \rfloor$.

Lemma 1. *If* $0 \le j \le \lfloor n/2 \rfloor$*, then*

$$m(j, S) = \min\{n + j - 2\ell_1, 2n - j - 2\ell_2\}. \tag{1}$$

Proof. Let W be a shortest $(0, j)$-walk in C_n that traverses all vertices in S. It is easy to see that W must be in either of the two forms W_1 and W_2 depicted in Fig. 3, respectively.

It is clear from Fig. 3 that $m(j, S) = n + j - 2\ell_1$ if $W = W_1$, $m(j, S) = 2n - j - 2\ell_2$ if $W = W_2$. The lemma follows. □

The relationship between the shortest $((o;\ 0), (x;\ j))$-paths in CCC_n and the shortest $(0, j)$-walks in C_n is stated in the following theorem.

Theorem 2. *Let* $x = x_0x_1 \dots x_{n-1} \in V(Q_n)$ *and* $S_x = \{i | x_i = 1, 0 \le i \le n-1\}$*. Then the distance between* $(o; 0)$ *and* $(x; j)$ *in* CCC_n

$$d_{CCC_n}((o; 0), (x; j)) = |S_x| + m(j, S_x). \tag{2}$$

Proof. Let P be a shortest path from $(o; 0)$ to $(x; j)$ in CCC_n. We first note that P contains at least $|S_x|$ cubic edges. In fact, following P from $(o; 0)$ to $(x; j)$ when a new vertex is visited by a cyclic edge its cubic coordinate does not change, while by a cubic edge its cubic coordinate changes one and only one bit. So the number of cubic edges in P is at least $|S_x|$.

We now prove that P contains at least $m(j, S_x)$ cyclic edges. To see this, let $C_n = (0, 1, \ldots, n-1)$ and $P = (x^0, e^1, x^1, e^2, \cdots, e^p, x^p)$, where $x^0 = (o; 0)$, $x^p = (x; j)$, p is the length of P, x^i is a vertex in P for each $i = 0, 1, \ldots, p$ and $e^i = (x^{i-1}, x^i)$ is an edge in P for each $i = 1, 2, \ldots, p$. We define a mapping f from $E(P)$ to C_n subject to:

(i) If $x^{i-1} = (x_0x_1 \ldots x_{n-1};\ k)$ and $x^i = (x_0x_1 \ldots x_{n-1};\ k\pm 1 \bmod n)$, i.e. $e^i = (x^{i-1}, x^i)$ is a cyclic edge, then $f(e^i) = (k, k\pm 1)$;
(ii) If $x^{i-1} = (x_0 \ldots x_k \ldots x_{n-1};\ k)$ and $x^i = (x_0 \ldots \overline{x_k} \ldots x_{n-1};\ k)$, i.e. $e^i = (x^{i-1}, x^i)$ is a cubic edge, then $f(e^i) = k$.

It is easy to show that the sequence of vertices and edges $(f(e^1), f(e^2), \ldots, f(e^p))$, which may not alternate, is a $(0, j)$-walk in C_n, say W, and that the number of cyclic edges on P is equal to the length of W. Moreover, observe that P contains all vertices with cyclic coordinates in S_x and for these vertices the bits of their cubic coordinates in the positions determined by S_x must be changed. Thus, the walk W contains all vertices in S_x, which implies that the length of W is at least $m(j, S_x)$. Thus, we have

$$d_{CCC_n}((o; 0), (x; j)) \geq |S_x| + m(j, S_x).$$

On the other hand, to show that the number of cyclic edges in P is at most $m(j, S_x)$, we construct a path from $(o; 0)$ and $(x; j)$ in CCC_n with length $|S_x| + m(j, S_x)$ based on a shortest $(0, j)$-walk $W = (i^0 = 0, i^1, i^2, \ldots, i^m)$ in C_n that traverses all vertices in S_x, where $0 \leq i^1, i^2, \ldots, i^m \leq n-1$ and $m = m(j, S_x)$. Such a path P is recursively constructed as follows.

1. $k = 0$. Let $P_0 = ((o; 0))$, $W_0 = (i^0 = 0)$.
2. Suppose that $P_k = ((o; 0), (x^1; p^1), \ldots, (x^k; p^k))$, $W_k = (0, i^1, \ldots, i^t)$, $t < m$, and $p^k = i^t$. Construct $P_{k+1} = ((o; 0), (x^1; p^1), \ldots, (x^k; p^k), (x^{k+1}; p^{k+1}))$ and W_{k+1} according to the following two cases.
   (i) $p^k \in S_x$ and $p^r (0 \leq r \leq k-1) \neq p^k$. Assume $x^k = x_0x_1 \ldots x_{n-1}$. Take $(x^{k+1}; p^{k+1}) = (x_0 \ldots x_{p^k-1}\overline{x_{p^k}}x_{p^k+1} \ldots x_{n-1}; p^k)$, and $W_{k+1} = W_k$.
   (ii) Otherwise, take $(x^{k+1}; p^{k+1}) = (x^k, i^{t+1})$ and $W_{k+1} = (0, i^1, \ldots, i^t, i^{t+1})$.
   In either case, we have $p^{k+1} = i^{t+1}$.

One can check that this process is finite and stops when $W_k = W$. Then let $P = P_k$, and we can verify that P is a path from $(o; 0)$ to $(x; j)$ of length $|S_x| + m(j, S_x)$. Thus, we have $d_{CCC_n}((o; 0), (x; j)) \leq |S_x| + m(j, S_x)$. The theorem follows. □

4. Main results

In this section, we will determine the vertex forwarding index of the cube-connected cycle CCC_n. The proof of our result depends strongly on the following lemma, which is due to Heydemann, Meyer and Sotteau [8].

Lemma 3. *If $G = (V, E)$ is a connected Cayley graph of order v, then for any x,*

$$\xi(G) = \sum_{y \in V} d(x, y) - (v - 1).$$

Since CCC_n is a connected Cayley graph of order $n2^n$, by Lemma 3, we have

$$\xi(CCC_n) = \sum_{\substack{x \in V(Q_n)\\ 0 \leq j \leq n-1}} dist((o; 0), (x; j)) - (n2^n - 1). \tag{3}$$

Let

$$R = \sum_{\substack{x \in V(Q_n)\\ 0 \leq j \leq n-1}} |S_x| \quad \text{and} \quad T = \sum_{\substack{x \in V(Q_n)\\ 0 \leq j \leq n-1}} m(j, S_x).$$

It follows from (3) and Theorem 2 that

$$\xi(CCC_n) = R + T - (n2^n - 1). \tag{4}$$

In the following lemmas, we derive the exact value of R and the asymptotic expression of T separately.

Lemma 4. *$R = n^2 2^{n-1}$ for any $n \geq 2$.*

Proof. Since Q_n is vertex-transitive, we have

$$R = \sum_{\substack{x \in V(Q_n)\\ 0 \leq j \leq n-1}} |S_x| = n \sum_{x \in V(Q_n)} |S_x| = n \sum_{k=0}^{n} k \binom{n}{k} = n^2 2^{n-1}$$

as required. □

To establish a lower bound on T, we recall the concept of the number of ordered partitions of a positive integer n.

An ordered partition of an integer n is an ordered collection of positive integers, called *parts*, whose sum is n. For example,

$$\{1, 1, 1, 1\}, \{2, 1, 1\}, \{1, 2, 1\}, \{1, 1, 2\}, \{3, 1\}, \{1, 3\}, \{2, 2\}, \{4\}$$

are eight ordered partitions of 4. Denote by $p(n)$ the number of ordered partitions of n and $p(n, \ell)$ the number of ordered partitions of n such that the maximum part is not more than ℓ. It is easy to prove that

$$p(n) = 2^{n-1}. \tag{5}$$

Lemma 5. *$p(n, \ell) \geq 2^{n-1} - (n-\ell)^2\, 2^{n-\ell-2}$ for any integers $n \geq 2$ and $\ell \leq n$.*

Proof. For given j and k with $1 \leq j \leq n$ and $0 \leq k \leq n-j$, let $A_{j,k}$ be the set of partitions of n in which there is a part j such that the sum of all parts before j is equal to k. For example, $\{1, 1, 2, 3\} \in A_{2,2} \cap A_{3,4}$ and $\{2, 1, 3, 1\} \in A_{3,3}$.

Clearly, if $k = 0$ or $n-j$, then j is fixed in the first or the last part of all partitions of n, and so $|A_{j,0}| = |A_{j,n-j}| = 2^{n-j-1}$. If $2 \leq k \leq n-j-1$, then $|A_{j,k}| = p(k) \cdot p(n-j-k)$. It follows from (5) that $|A_{j,k}| = 2^{k-1} \cdot 2^{n-j-1-k} = 2^{n-j-2}$. Thus, we have that

$$|A_{j,k}| = \begin{cases} 2^{n-j-1} & \text{for } k = 0 \text{ or } n-j; \\ 2^{n-j-2} & \text{otherwise,} \end{cases}$$

from which we have that

$$|A_{j,k}| \leq |A_{j,0}| = 2^{n-j-1} \quad \text{for any } j \text{ and } k. \tag{6}$$

Obviously, each partition of n whose largest part is larger than ℓ is in $\bigcup_{\substack{\ell+1\leq j\leq n\\ 0\leq k\leq n-j}} A_{j,k}$. It follows from (6) that

$$\begin{aligned} p(n, \ell) &\geq 2^{n-1} - \left| \bigcup_{\substack{\ell+1\leq j\leq n\\ 0\leq k\leq n-j}} A_{j,k} \right| \\ &\geq 2^{n-1} - (n-\ell)^2\, |A_{\ell+1,0}| \\ &= 2^{n-1} - (n-\ell)^2\, 2^{n-\ell-2} \end{aligned}$$

as required. □

Corollary 6. *Let $\ell = \lfloor \log_2^2 n \rfloor$. Then $p(n, \ell) \geq 2^{n-1}(1 - n^{2-\log_2 n})$.*

Proof. By Lemma 5, we immediately have that

$$\begin{aligned} p(n, \ell) &\geq 2^{n-1} - (n-\ell)^2 2^{n-\ell-2} \\ &\geq 2^{n-1} - n^2 2^{n-\ell-2} \\ &= 2^{n-1}(1 - n^2 2^{-\ell-1}) \\ &\geq 2^{n-1}(1 - n^{2-\log_2 n}) \end{aligned}$$

as required. □

Lemma 7. *$T \geq \frac{5}{4} n^2 2^n (1 - o(1))$.*

Proof. By symmetry, we have

$$T = \sum_{\substack{x\in V(Q_n)\\ 0\leq j\leq n-1}} m(j, S_x) \geq 2 \sum_{\substack{x\in V(Q_n)\\ 1\leq j\leq \lfloor (n-1)/2\rfloor}} m(j, S_x) = 2 \sum_{\substack{S\subseteq\{0,1,\ldots,n-1\}\\ 1\leq j\leq \lfloor (n-1)/2\rfloor}} m(j, S). \tag{7}$$

Let $A = \lfloor \log_2^2 n \rfloor$. Recall that ℓ_1 and ℓ_2 are gaps defined in Section 3, and $\ell_1, \ell_2 \geq 1$. For any $S \subseteq \{0, 1, \ldots, n-1\}$ and any integer j with $A \leq j \leq \lfloor n/2 \rfloor - A$, if

$$\ell_1 \leq \lfloor \log_2^2 j \rfloor \quad \text{and} \quad \ell_2 \leq \lfloor \log_2^2 (n-j) \rfloor, \tag{8}$$

then

$$\ell_2 - \ell_1 < \ell_2 \leq \lfloor \log_2^2 (n-j) \rfloor < A \leq \lfloor n/2 \rfloor - j \leq n/2 - j,$$

and so

$$n + j - 2\ell_1 < 2n - j - 2\ell_2. \tag{9}$$

It follows from (1) and (9) that $m(j, S) = n + j - 2\ell_1 \geq n + j - 2A$, that is,

$$m(j, S) \geq n + j - 2A. \tag{10}$$

For a given j, let $b(j)$ be the number of subsets S such that ℓ_1 and ℓ_2 satisfy the condition (8). Then

$$b(j) = 4p(j, L_1) \cdot p(n - j, L_2),$$

where $L_1 = \lfloor \log_2^2 j \rfloor$ and $L_2 = \lfloor \log_2^2 (n - j) \rfloor$, and 4 is the four choices of whether 0 and (or) j are (is) in S. By Corollary 6, we have that

$$\begin{aligned} b(j) &= 4p(j, L_1) \cdot p(n - j, L_2) \\ &\geq 4 \cdot 2^{j-1}(1 - j^{2-\log_2 j}) \cdot 2^{n-j-1}(1 - (n-j)^{2-\log_2 (n-j)}) \\ &= 2^n(1 - o(1)) \end{aligned} \tag{11}$$

when $A \leq j \leq \lfloor n/2 \rfloor - A$. It follows from (7), (10) and (11) that

$$\begin{aligned} T &\geq 2 \sum_{\substack{S \subseteq \{0,1,\ldots,n-1\} \\ 1 \leq j \leq \lfloor (n-1)/2 \rfloor}} m(j, S) \\ &\geq 2 \sum_{j=A}^{\lfloor \frac{n}{2} \rfloor - A} (n + j - 2A) \cdot 2^n (1 - o(1)) \\ &= 2^{n+1}(1 - o(1)) \cdot \left(\left(\left\lfloor \frac{n}{2} \right\rfloor - 2A + 1 \right) (n - 2A) + \left\lfloor \frac{n}{4} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 2A + 1 \right) \right) \\ &\geq 2^{n+1}(1 - o(1)) \cdot \left(\left(\frac{n}{2} - 2A \right) \left(\frac{5}{4} n - 2A - 1 \right) \right) \\ &= \frac{5}{4} n^2 2^n (1 - o(1)) \end{aligned}$$

as required. □

Lemma 8. $T = \frac{5}{4} n^2 2^n (1 - o(1))$.

Proof. By Lemma 7, it is sufficient to prove that $T \leq \frac{5}{4} n^2 2^n (1 + o(1))$. In fact, from formula (1), we have $m(j, S) \leq n + j$ if $0 \leq j \leq \lceil n/2 \rceil$. Thus,

$$\begin{aligned} T &= \sum_{\substack{x \in V(Q_n) \\ 0 \leq j \leq n-1}} m(j, S_x) = \sum_{\substack{S \subseteq \{0,1,\ldots,n-1\} \\ 0 \leq j \leq n-1}} m(j, S) \\ &\leq 2 \sum_{\substack{S \subseteq \{0,1,\ldots,n-1\} \\ 0 \leq j \leq \lceil n/2 \rceil}} m(j, S) \leq 2 \sum_{\substack{S \subseteq \{0,1,\ldots,n-1\} \\ 0 \leq j \leq \lceil n/2 \rceil}} (n + j) \\ &= 2^{n+1} \sum_{j=0}^{\lceil n/2 \rceil} (n + j) = \frac{5}{4} n^2 2^n (1 + o(1)) \end{aligned}$$

as required. The lemma follows. □

Theorem 9. *$\xi(CCC_n) = \frac{7}{4} n^2 2^n (1 - o(1))$ for any integer $n \geq 2$.*

Proof. From (4), Lemmas 4 and 8, we immediately have that

$$\begin{aligned} \xi(CCC_n) &= R + T - (n2^n - 1) \\ &= n^2 2^{n-1} + \frac{5}{4} n^2 2^n (1 - o(1)) - (n2^n - 1) \\ &= \frac{7}{4} n^2 2^n (1 - o(1)) \end{aligned}$$

as required and so the theorem follows. □

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